

Curvature estimates for properly immersed ϕ_h -bounded submanifolds

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Abstract

Jorge-Koutrofotis [14] & Pigola-Rigoli-Setti [23] proved sharp sectional curvature estimates for extrinsically bounded submanifolds. Alias, Bessa and Montenegro in [2], showed that these estimates hold on properly immersed cylindrically bounded submanifolds. On the other hand, in [1], Alias, Bessa and Dajczer proved sharp mean curvature estimates for properly immersed cylindrically bounded submanifolds. In this paper we prove these sectional and mean curvature estimates for a larger class of submanifolds, the properly immersed ϕ -bounded submanifolds, Thms. 2.3 & 2.5. These ideas, in fact, we prove stronger forms of these estimates, see the results in section 4.

keywords: Curvature estimates, ϕ -bounded submanifolds, Omori-Yau pairs, Omori-Yau maximum principle.

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1 Introduction

The classical isometric immersion problem asks whether there exists an isometric immersion $\varphi: M \rightarrow N$ for given Riemannian manifolds M and N of dimension m and n respectively, with $m < n$. The model result for this type of problem is the celebrated Efimov-Hilbert Theorem [11], [13] that says that there is no isometric immersion of a geodesically complete surface M with sectional curvature $K_M \leq -\delta^2 < 0$ into \mathbb{R}^3 , $\delta \in \mathbb{R}$. On the other hand, the Nash Embedding Theorem shows that there is always an isometric embedding into the Euclidean n -space \mathbb{R}^n provided the codimension $n - m$ is sufficiently large, see [17]. For small codimension, meaning in this paper that $n - m \leq m - 1$, the answer in general depends on the geometries of M and N . For instance, a classical result of C. Tompkins [27] states that a compact, flat, m -dimensional Riemannian manifold can not be isometrically immersed into \mathbb{R}^{2m-1} . C. Tompkins's result was extended in a series of papers, by

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Chern and Kuiper [9], Moore [16], O'Neill [19], Otsuki [20] and Stiel [25], whose results can be summarized in the following theorem.

Theorem 1.1 (C. Tompkins et al.) *Let $\varphi: M \rightarrow N$ be an isometric immersion of compact Riemannian m -manifold M into a Cartan-Hadamard n -manifold N with small codimension $n - m \leq m - 1$. Then the sectional curvatures of M and N satisfy*

$$\sup_M K_M > \inf_N K_N. \quad (1)$$

L. Jorge and D. Koutrofiotis [14], considered complete extrinsically bounded¹ submanifolds with scalar curvature bounded from below and proved the curvature estimates (3). Pigola, Rigoli and Setti [23] proved an all general and abstract version of the Omori-Yau maximum principle [8], [28] and in consequence they were able to extend Jorge-Koutrofiotis' Theorem to complete m -submanifolds M immersed into regular balls of any Riemannian n -manifold N with scalar curvature bounded below as $s_M \geq -c \cdot \rho_M^2 \cdot \prod_{j=1}^k \left(\log^{(j)}(\rho_M) \right)^2$, $\rho_M \gg 1$.

Their version of Jorge-Koutrofiotis Theorem is the following.

Theorem 1.2 (Jorge-Koutrofiotis & Pigola-Rigoli-Setti) *Let $\varphi: M \rightarrow N$ be an isometric immersion of a complete Riemannian m -manifold M into a n -manifold N , with $n - m \leq m - 1$, with $\varphi(M) \subset B_N(r)$, where $B_N(r)$ is a regular geodesic ball of N . If the scalar curvature of M satisfies*

$$s_M \geq -c \cdot \rho_M^2 \cdot \prod_{j=1}^k \left(\log^{(j)}(\rho_M) \right)^2, \quad \rho_M \gg 1, \quad (2)$$

for some constant $c > 0$ and some integer $k \geq 1$, where ρ_M is the distance function on M to a fixed point and $\log^{(j)}$ is the j -th iterate of the logarithm. Then

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N, \quad (3)$$

where $b = \sup_{B_N(r)} K_N^{\text{rad}} \leq b$

$$C_b(t) = \begin{cases} \sqrt{b} \cot(\sqrt{b}t) & \text{if } b > 0 \text{ and } 0 < t < \pi/2\sqrt{b} \\ 1/t & \text{if } b = 0 \text{ and } t > 0 \\ \sqrt{-b} \coth(\sqrt{-b}t) & \text{if } b < 0 \text{ and } t > 0. \end{cases} \quad (4)$$

¹Meaning: immersed into regular geodesic balls of a Riemannian manifold.

Remark 1.3 If $B(r) \subset \mathbb{N}^n(b)$ is a geodesic ball of radius r in the simply connected space form of sectional curvature b , $\partial B(r)$ its boundary and $\varphi: \partial B(r - \varepsilon) \rightarrow B(r)$ is the canonical immersion, where $\varepsilon > 0$ is small, then we have

$$\sup_M K_M = K_{\partial B(r-\varepsilon)} = \begin{cases} b/\sin^2(\sqrt{b}(r-\varepsilon)) & \text{if } b > 0 \\ 1/(r-\varepsilon)^2 & \text{if } b = 0 \\ -b/\sinh^2(\sqrt{-b}(r-\varepsilon)) & \text{if } b < 0. \end{cases}$$

Therefore, $\sup_M K_M - [C_b^2(r) + \inf K_{\mathbb{N}^n(b)}] = [C_b^2(r - \varepsilon) - C_b^2(r)] \rightarrow 0$ as $\varepsilon \rightarrow 0$, showing that the inequality (3) is sharp.

Remark 1.4 One may assume without loss of generality that $\sup_M K_M < \infty$. This together with the scalar curvature bounds (2) implies that

$$K_M \geq -c^2 \cdot \rho_M^2 \cdot \prod_{j=1}^k \left(\log^{(j)}(\rho_M) \right)^2, \rho_M \gg 1$$

for some positive constant $c > 0$. This curvature lower bound implies that M is stochastically complete, which it is equivalent to the fact that M hold the weak maximum principle, (a weaker form of Omori-Yau maximum principle, see details in [22]), and that is enough to reproduce Jorge-Koutrofitis original proof of the curvature estimate (3).

Recently, Alias, Bessa and Montenegro [2] extended Theorem 1.2 to the class of cylindrically bounded, properly immersed submanifolds, where an isometric immersion $\varphi: M \hookrightarrow N \times \mathbb{R}^\ell$ is cylindrically bounded if $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$. Here $B_N(r)$ is a geodesic ball in N of radius $r > 0$. They proved the following theorem.

Theorem 1.5 (Alias-Bessa-Montenegro) Let $\varphi: M \rightarrow N \times \mathbb{R}^\ell$ be a cylindrically bounded isometric immersion, $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, where $B_N(r)$ is a regular geodesic ball of N and $b = \sup_{B_N(r)} K^{\text{rad}}$. Let $\dim(M) = m$, $\dim(N) = n - \ell$ and assume that $n - m \leq m - \ell - 1$. If either

i. the scalar curvature of M is bounded below as (2), or

ii. the immersion φ is proper and

$$\sup_{\varphi^{-1}(B_N(r) \times \partial B_{\mathbb{R}^\ell}(t))} \|\alpha\| \leq \sigma(t), \quad (5)$$

where α is the second fundamental form of φ and $\sigma: [0, +\infty) \rightarrow \mathbb{R}$ is a positive function satisfying $\int_0^{+\infty} dt/\sigma(t) = +\infty$, then

$$\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N. \quad (6)$$

Remark 1.6 *The idea is to show that the hypotheses, in both items i. & ii. implies that M is stochastically complete, then Remark 1.4 applies.*

In the same spirit, Alias, Bessa and Dajczer [1], had proved the following mean curvature estimates for cylindrically bounded submanifolds properly immersed into $N \times \mathbb{R}^\ell$ immersed submanifolds.

Theorem 1.7 (Alias-Bessa-Dajczer) *Let $\varphi: M \rightarrow N \times \mathbb{R}^\ell$ be a cylindrically bounded isometric immersion, $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, where $B_N(r)$ is a regular geodesic ball of N and $b = \sup K_{B_N(r)}^{\text{rad}}$. Here M and N are complete Riemannian manifolds of dimension m and $n - \ell$ respectively, satisfying $m \geq \ell + 1$. If the immersion φ is proper, then*

$$\sup_M |H| \geq (m - \ell) \cdot C_b(r). \quad (7)$$

2 Main results

The purpose of this paper is to extend these curvature estimates to a larger class of submanifolds, precisely, the properly immersed ϕ -bounded submanifolds. To describe this class we need to introduce few preliminaries.

2.1 ϕ -bounded submanifolds

Consider $G \in C^\infty([0, \infty))$ satisfying

$$G_- \in L^1(\mathbb{R}^+), \quad t \int_t^{+\infty} G_-(s) ds \leq \frac{1}{4} \quad \text{on } \mathbb{R}^+, \quad (8)$$

and h the solution of the following differential equation

$$\begin{cases} h''(t) - G(t)h(t) = 0, \\ h(0) = 0, \quad h'(0) = 1. \end{cases} \quad (9)$$

In [6, Prop. 1.21], it is proved that the solution h and its derivative h' are positive in $\mathbb{R}^+ = (0, \infty)$, provided G satisfies (8) and furthermore $h \rightarrow +\infty$ whenever the stronger condition

$$G(s) \geq -\frac{1}{4s^2} \quad \text{on } \mathbb{R}^+ \quad (10)$$

holds. Define $\phi_h \in C^\infty([0, \infty))$ by

$$\phi_h(t) = \int_0^t h(s) ds. \quad (11)$$

Since h is positive and increasing in \mathbb{R}^+ , we have that $\lim_{t \rightarrow \infty} \phi_h(t) = +\infty$. Moreover, ϕ_h satisfies the differential equation

$$\phi_h''(t) - \frac{h'}{h}(t)\phi_h'(t) = 0$$

for all $t \in [0, \infty)$.

Notation. In this paper, N will always be a complete Riemannian manifold with a distinguished point z_0 and radial sectional curvatures along the minimal geodesic issuing from z_0 bounded above by $K_N^{\text{rad}}(z) \leq -G(\rho_N(z))$, where G satisfies the conditions (8). Let h be the solution of (9) associated to G and $\phi_h = \int h(s)ds$. Finally, $\rho_N(z) = \text{dist}_N(z_0, z)$ will be the distance function on N . For any given complete Riemannian manifold (L, y_0) with a distinguished point y_0 and radial sectional curvature² bounded below ($K_L^{\text{rad}} \geq -\Lambda^2$) and $\varepsilon \in (0, 1)$ consider the subset $\Omega_{\phi_h}(\varepsilon) \subset N \times L$ given by

$$\Omega_{\phi_h}(\varepsilon) = \{(x, y) \in N \times L : \phi_h(\rho_N(x)) \leq \log(\rho_L(y) + 1)^{1-\varepsilon}\}.$$

Here $\rho_L(y) = \text{dist}_L(y_0, y)$, $y_0 \in L$.

Definition 2.1 An isometric immersion $\varphi: M \rightarrow N \times L$ of a Riemannian manifold M into the product $N \times L$ is said to be ϕ_h -bounded if there exists a compact $K \subset M$ and $\varepsilon \in (0, 1)$ such that $\varphi(M \setminus K) \subset \Omega_h(\varepsilon)$.

Remark 2.2 The class of ϕ -bounded submanifolds contains the class of cylindrically bounded submanifolds.

2.2 Curvature estimates for ϕ -bounded submanifolds

In this section, we extend the cylindrically bounded version of Jorge-Koutrofiotis's Theorem, Thm. 1.5-ii. and the mean curvature estimates of Thm. 1.7 to the class of ϕ_h -bounded properly immersed submanifolds. These extensions are done in two ways. First: the class we consider is larger than the class of cylindrically bounded submanifolds. Second: there are no requirements on the growth on the second fundamental form as in Thm. 1.5. We also should observe that although ϕ -bounded properly immersed submanifolds, $(\varphi: M \rightarrow N \times L)$ are stochastically complete, provided L has an Omori-Yau pair, see Section 4, we do not need that to prove the following result.

²Along the geodesics issuing from y_0 .

Theorem 2.3 *Let $\varphi: M \rightarrow N^{n-\ell} \times L^\ell$ be a ϕ_h -bounded isometric immersion of a complete Riemannian m -manifold M with $n - m \leq m - \ell - 1$. If φ is proper and $-G \leq b \leq 0$ then*

$$\sup_M K_M \geq |b| + \inf_N K_N. \quad (12)$$

With strict inequality $\sup_M K_M > \inf_N K_N$ if $b = 0$.

Corollary 2.4 *Let $\varphi: M \rightarrow N^{n-\ell} \times L^\ell$ be a properly immersed, cylindrically bounded submanifold, $\varphi(M) \subset B_N(r) \times L^\ell$, where $B_N(r)$ is a regular geodesic ball of N . Suppose that $n - m \leq m - \ell - 1$. Then the sectional curvature of M satisfies the following inequality*

$$\sup_M K_M \geq C_b^2(r) + \inf_N K_N, \quad (13)$$

where $b = \sup_{B_N(r)} K_N^{rad}$ and C_b is defined in (4).

Our next main result extends the mean curvature estimates (7) to ϕ -bounded submanifolds.

Theorem 2.5 *Let $\varphi: M \rightarrow N^{n-\ell} \times L^\ell$ be a ϕ_h -bounded isometric immersion of a complete Riemannian m -manifold M with $m \geq \ell + 1$. If φ is proper then the mean curvature vector $H = \text{tr } \alpha$ of φ satisfies*

$$\sup_M |H| \geq (m - \ell) \cdot \inf_{r \in [0, \infty)} \frac{h'}{h}(r). \quad (14)$$

If $-G \leq b \leq 0$ then

$$\sup_M |H| \geq (m - \ell) \cdot \sqrt{|b|}. \quad (15)$$

With strict inequality $\sup_M |H| > 0$ if $b = 0$.

3 Proof of the main results

3.1 Basic results

Let M and W be Riemannian manifolds of dimension m and n respectively and let $\varphi: M \rightarrow W$ be an isometric immersion. For a given function $g \in C^\infty(W)$ set $f = g \circ \varphi \in C^\infty(M)$. Since

$$\langle \text{grad}_M f, X \rangle = \langle \text{grad}_W g, X \rangle$$

for every vector field $X \in TM$, we obtain

$$\text{grad}_W g = \text{grad}_M f + (\text{grad}_W g)^\perp$$

according to the decomposition $TW = TM \oplus T^\perp M$. An easy computation using the Gauss formula gives the well-known relation (see e.g. [14])

$$\text{Hess}_M f(X, Y) = \text{Hess}_W g(X, Y) + \langle \text{grad}_W g, \alpha(X, Y) \rangle \quad (16)$$

for all vector fields $X, Y \in TM$, where α stands for the second fundamental form of φ . In particular, taking traces with respect to an orthonormal frame $\{e_1, \dots, e_m\}$ in TM yields

$$\Delta_M f = \sum_{i=1}^m \text{Hess}_W g(e_i, e_i) + \langle \text{grad}_W g, H \rangle. \quad (17)$$

where $H = \sum_{i=1}^m \alpha(e_i, e_i)$.

In the sequel, we will need the following well known results, see the classical Greene-Wu [12] for the Hessian Comparison Theorem and Pigola-Rigoli-Setti's "must looking at" book [24, Lemma 2.13], see also [26], [6, Thm.1.9] for the Sturm Comparison Theorem.

Theorem 3.1 (Hessian Comparison Thm.) *Let W be a complete n -manifold and $\rho_W(x) = \text{dist}_W(x_0, x)$, $x_0 \in W$ fixed. Let $D_{x_0} = W \setminus (\{x_0\} \cup \text{cut}(x_0))$ be the domain of normal geodesic coordinates at x_0 . Let $G \in C^0([0, \infty))$ and let h be the solution of (9). Let $[0, R)$ be the largest interval where $h > 0$. Then*

i. *If the radial sectional curvatures along the geodesics issuing from x_0 satisfies*

$$K_W^{\text{rad}} \geq -G(\rho_W), \text{ in } B_W(R)$$

then

$$\text{Hess}_W \rho \leq \frac{h'}{h}(\rho_W) [\langle, \rangle - d\rho \otimes d\rho] \text{ on } D_{x_0} \cap B_W(R)$$

ii. *If the radial sectional curvatures along the geodesics issuing from x_0 satisfy*

$$K_W^{\text{rad}} \leq -G(\rho_W), \text{ in } B_W(R)$$

then

$$\text{Hess}_W \rho_W \geq \frac{h'}{h}(\rho) [\langle, \rangle - d\rho \otimes d\rho] \text{ on } D_{x_0} \cap B_W(R)$$

Lemma 3.2 (Sturm Comparison Thm.) *Let $G_1, G_2 \in L_{\text{loc}}^\infty(\mathbb{R})$, $G_1 \leq G_2$ and h_1 and h_2 solutions of the following problems:*

$$a.) \begin{cases} h_1''(t) - G_1(t)h_1(t) & \leq 0 \\ h_1(0) = 0, \quad h_1'(0) & > 0 \end{cases} \quad b.) \begin{cases} h_2''(t) - G_2(t)h_2(t) & \geq 0 \\ h_2(0) = 0, \quad h_2'(0) & > h_1'(0), \end{cases} \quad (18)$$

and let $I_1 = (0, S_1)$ and $I_2 = (0, S_2)$ be the largest connected intervals where $h_1 > 0$ and $h_2 > 0$ respectively. Then

1. $S_1 \leq S_2$. And on I_1 , $\frac{h'_1}{h_1} \leq \frac{h'_2}{h_2}$ and $h_1 \leq h_2$.

2. If $h_1(t_o) = h_2(t_o)$, $t_o \in I_1$ then $h_1 \equiv h_2$ on $(0, t_o)$.

For a more detailed Sturm Comparison Theorem one should consult the beautiful book [24, Chapter 2.]. If $-G = b \in \mathbb{R}$ then the solution of $h''_b(t) - G \cdot h_b(t) = 0$ with $h_b(0) = 0$ and $h'_b(0) = 1$ is given by

$$h_b(t) = \begin{cases} \frac{1}{\sqrt{-b}} \cdot \sinh(\sqrt{-b}t) & \text{if } b < 0 \\ t & \text{if } b = 0 \\ \frac{1}{\sqrt{b}} \cdot \sin(\sqrt{b}t) & \text{if } b > 0. \end{cases}$$

In particular, if the radial sectional curvatures along the geodesics issuing from x_0 satisfy $K_w^{rad}(x) \leq -G(\rho_w(x)) \leq b$, $x \in B_w(R) = \{x, \text{dist}_w(x_0, x) = \rho_w(x) < R\}$, then the solution h of (9), satisfies $(h'/h)(t) \geq (h'_b/h_b)(t) = C_b(t)$, $t \in (0, R)$, $R < \pi/2\sqrt{b}$, if $b > 0$. Therefore, $\text{Hess}_w \rho_w \geq C_b(\rho_w) [\langle, \rangle - d\rho_w \oplus d\rho_w]$. Likewise, if $K_w^{rad}(x) \geq -G(\rho_w(x)) \geq b$, $x \in B_w(R)$ then $(h'/h)(t) \leq C_b(t)$, $t \in (0, R)$ and $\text{Hess}_w \rho_w \leq C_b(\rho_w) [\langle, \rangle - d\rho_w \oplus d\rho_w]$.

3.2 Proof of Theorem 2.3.

Assume without loss that there exists a $x_0 \in M$ such that $\varphi(x_0) = (z_0, y_0) \in N \times L$, z_0, y_0 the distinguished points of N and L . For each $x \in M$, let $\varphi(x) = (z(x), y(x))$. Define $g: N \times L \rightarrow \mathbb{R}$ by $g(z, y) = \phi_h(\rho_N(z)) + 1$, recalling that $\phi_h(t) = \int_0^t h(s)ds$, and define $f = g \circ \varphi: M \rightarrow \mathbb{R}$ by $f(x) = g(\varphi(x)) = \phi_h(\rho_N(z(x))) + 1$. For each $k \in \mathbb{N}$, set $g_k(x) = f(x) - \frac{1}{k} \cdot \log(\rho_L(y(x)) + 1)$. Observe that $g_k(x_0) = 1$ for all k , since $\rho_N(z_0) = \rho_L(y_0) = 0$. First, let us prove the item i.

If $x \rightarrow \infty$ in M then $\varphi(x) \rightarrow \infty$ in $N \times L$ since φ is proper. On the other hand, $\varphi(M \setminus K) \subset \Omega_h(\varepsilon)$ for some compact $K \subset M$ and $\varepsilon \in (0, 1)$. This implies that $y(x) \rightarrow \infty$ in L and

$$\frac{g_k(x)}{\log(\rho_L(y(x)) + 1)} = \frac{f(x)}{\log(\rho_L(y(x)) + 1)} - \frac{1}{k} < \frac{1}{\log(\rho_L(y(x)) + 1)^\varepsilon} - \frac{1}{k} < 0$$

for $\rho_M(x) \gg 1$. This implies that $g_k(x) < 0$ for $\rho_M(x) \gg 1$. Therefore each g_k reach a maximum at a point $x_k \in M$. This yields a sequence $\{x_k\} \subset M$ so that $\text{Hess}_M g_k(x_k)(X, X) \leq 0$ for all $X \in T_{x_k} M$, this is, $\forall X \in T_{x_k} M$

$$\text{Hess}_M f(x_k)(X, X) \leq \frac{1}{k} \cdot \text{Hess}_M \log(\rho_L(y(x_k)) + 1)(X, X). \quad (19)$$

Observe that $\log(\rho_L(y(x_k)) + 1) = \log(\rho_L \circ \pi_L + 1) \circ \varphi(x_k)$, $\pi_L: N \times L \rightarrow L$ the projection on the second factor, thus the right hand side of (19), using the formula (16), is given by

$$\begin{aligned} \text{Hess}_M \log(\rho_L(y(x_k)) + 1)(X, X) &= \text{Hess}_{N \times L} \log(\rho_L \circ \pi_L + 1)(\varphi(x_k))(X, X) \\ &+ \langle \text{grad}_{N \times L} \log(\rho_L \circ \pi_L + 1), \alpha(X, X) \rangle \end{aligned} \quad (20)$$

Where α is the second fundamental form of φ . For simplicity, set $\psi(t) = \log(t + 1)$, $z_k = z(x_k)$, $y_k = y(x_k)$, $s_k = \rho_N(z_k)$ and $t_k = \rho_L(y_k)$. Decomposing $X \in TM$ as $X = X^N + X^L \in TN \oplus TL$, we see that the first term of the right hand side of (20) is

$$\begin{aligned} \text{Hess}_{N \times L} \psi \circ \rho_L \circ y(x_k)(X, X) &= \psi''(t_k) |X^L|^2 + \psi'(t_k) \text{Hess}_L \rho_L(y_k)(X, X) \\ &\leq \psi''(t_k) |X^L|^2 + C_{-\Lambda^2}(t_k) \frac{|X^N|^2}{(t_k + 1)} \\ &\leq C_{-\Lambda^2}(t_k) \frac{|X^N|^2}{(t_k + 1)}, \end{aligned} \quad (21)$$

since $\text{Hess}_L \rho_L(y_k)(X, X) \leq C_{-\Lambda^2}(t_k) |X^N|^2$ (by Theorem 3.1) and $\psi'' \leq 0$.

The second term of the right hand side of (20) is

$$\begin{aligned} \langle \text{grad}_{N \times L} \psi \circ \rho_L \circ y(x_k), \alpha(X, X) \rangle &= \psi'(t_k) \langle \text{grad}_L \rho_L(y_k), \alpha(X, X) \rangle \\ &\leq \frac{1}{(t_k + 1)} \|\alpha\| \cdot |X|^2 \end{aligned} \quad (22)$$

From (21) and (22) we have the following

$$\text{Hess}_M \psi \circ \rho_L \circ y(x_k)(X, X) \leq \frac{C_{-\Lambda^2}(t_k) + \|\alpha\|}{(t_k + 1)} \cdot |X|^2 \quad (23)$$

And from (19) and (23) we have that

$$\text{Hess}_M f(x_k)(X, X) \leq \frac{1}{k} \frac{(C_{-\Lambda^2}(t_k) + \|\alpha\|)}{(t_k + 1)} |X|^2 \quad (24)$$

We will compute the left hand side of (19). Using the formula (16) again we have

$$\text{Hess}_M f(x_k) = \text{Hess}_{N \times L} g(\varphi(x_k)) + \langle \text{grad}_{N \times L} g, \alpha \rangle \quad (25)$$

Recalling that $f = g \circ \varphi$ and g is given by $g(z, y) = \phi_h(\rho_N(z))$, where ϕ_h is defined in (11) and $\rho_N(z) = \text{dist}_N(z_0, z)$. Let us consider an orthonormal basis (26)

$$\{\overbrace{\text{grad } \rho_N, \partial/\partial \theta_1, \dots, \partial/\partial \theta_{n-\ell-1}}^{\in TN}, \overbrace{\partial/\partial \gamma_1, \dots, \partial/\partial \gamma_\ell}^{\in TL}\} \quad (26)$$

for $T_{\varphi(x_k)}(N \times L)$. Thus if $X \in T_{x_k}M$, $|X| = 1$, we can decompose

$$X = a \cdot \text{grad } \rho_N + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial/\partial \theta_j + \sum_{i=1}^{\ell} c_i \cdot \partial/\partial \gamma_i$$

with $a^2 + \sum_{j=1}^{n-\ell-1} b_j^2 + \sum_{i=1}^{\ell} c_i^2 = 1$. Recalling that $s_k = \rho_N(z(x_k))$, we can see that the first term of the right hand side of (25)

$$\begin{aligned} \text{Hess}_{N \times L} g(\varphi(x))(X, X) &= \phi_h''(s_k) \cdot a^2 + \phi_h'(s_k) \sum_{j=1}^{n-\ell-1} b_j^2 \cdot \text{Hess } \rho_N(z_k) \left(\frac{\partial}{\partial \theta_j}, \frac{\partial}{\partial \theta_j} \right) \\ &\geq \phi_h''(s_k) \cdot a^2 + \phi_h'(s_k) \sum_{j=1}^{n-\ell-1} b_j^2 \cdot \frac{h'}{h}(s_k) \\ &= \phi_h''(s_k) \cdot a^2 + (1 - a^2 - \sum_{i=1}^{\ell} c_i^2) \cdot \phi_h'(s_k) \cdot \frac{h'}{h}(s_k) \\ &= \left[\overbrace{(\phi_h'' - \frac{h'}{h} \cdot \phi_h')}^{\equiv 0} a^2 + (1 - \sum_{i=1}^{\ell} c_i^2) \cdot \phi_h' \cdot \frac{h'}{h} \right] (s_k) \\ &= (1 - \sum_{i=1}^{\ell} c_i^2) \cdot \phi_h'(s_k) \cdot \frac{h'}{h}(s_k) \end{aligned}$$

Thus

$$\text{Hess}_{N \times L} g(\varphi(x))(X, X) \geq (1 - \sum_{i=1}^{\ell} c_i^2) \cdot \phi_h'(s_k) \cdot \frac{h'}{h}(s_k). \quad (27)$$

The second term of the right hand side of (25) is the following

$$\begin{aligned} \langle \text{grad}_{N \times L} g, \alpha(X, X) \rangle &= \phi_h'(s_k) \langle \text{grad}_N \rho_N(z_k), \alpha(X, X) \rangle \\ &\geq -\phi_h'(s_k) |\alpha(X, X)| \end{aligned} \quad (28)$$

From (25), (27), (28) we have that,

$$\text{Hess}_M f(x_k)(X, X) \geq \left[\left(1 - \sum_{i=1}^{\ell} c_i^2\right) \cdot \frac{h'}{h}(s_k) - |\alpha(X, X)| \right] \phi'_b(s_k) \quad (29)$$

Recall that $n + \ell \leq 2m - 1$. This dimensional restriction implies that $m \geq \ell + 2$, since $n \geq m + 1$. Therefore, for every $x \in M$ there exists a sub-space $V_x \subset T_x M$ with $\dim(V_x) \geq (m - \ell) \geq 2$ such that $V \perp TL$, this is equivalent to $c_i = 0$. If we take any $X \in V_{x_k} \subset T_{x_k} M$, $|X| = 1$ we have by (29) that

$$\frac{(C_{-\Lambda^2}(t_k) + |\alpha(X, X)|)}{k(t_k + 1)} \geq \text{Hess}_M f(x_k)(X, X) \geq \left[\frac{h'}{h}(s_k) - |\alpha(X, X)| \right] \phi'_h(s_k)$$

Thus, reminding that $\phi'_h = h$,

$$\left[\frac{1}{k(t_k + 1)} + h(s_k) \right] |\alpha(X, X)| \geq h'(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1)} \quad (30)$$

Since $-G \leq b \leq 0$, we have by Lemma 3.2 (Sturm's argument) that the solution h of (9) satisfies $(h'/h)(t) \geq C_b(t) > \sqrt{|b|}$ and that $h(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, where C_b is defined in (4). Let us assume that $x_k \rightarrow \infty$ in M , (the case $\rho_M(x_k) \leq C^2 < \infty$ will be considered later), then $s_k \rightarrow \infty$ as well as $t_k \rightarrow \infty$. Thus from (30), for sufficiently large k , we have at $\varphi(x_k)$ that

$$\begin{aligned} \left[\frac{1}{k(t_k + 1)h(s_k)} + 1 \right] |\alpha(X, X)| &\geq \frac{h'(s_k)}{h(s_k)} - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1)h(s_k)} \\ &\geq C_b(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1)h(s_k)} \\ &> 0 \end{aligned} \quad (31)$$

Thus, at x_k and $X \in T_{x_k} M$ with $|X| = 1$ we have

$$|\alpha(X, X)| \geq \left[C_b(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1)h(s_k)} \right] \left[\frac{1}{k(t_k + 1)h(s_k)} + 1 \right]^{-1} > 0. \quad (32)$$

We will need the following lemma known as Otsuki's Lemma [15, p.28].

Lemma 3.3 (Otsuki) *Let $\beta: \mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathbb{R}^d$, $d \leq q - 1$, be a symmetric bilinear form satisfying $\beta(X, X) \neq 0$ for $X \neq 0$. Then there exists linearly independent vectors X, Y such that $\beta(X, X) = \beta(Y, Y)$ and $\beta(X, Y) = 0$.*

The *horizontal* subspace V_{x_k} has dimension $\dim(V_{x_k}) \geq m - \ell \geq 2$. Thus, by the inequality (32) and $n - m \leq m - \ell - 1 \leq \dim(V_{x_k}) - 1$, we may apply Otsuki's Lemma to $\alpha(x_k): V_{x_k} \times V_{x_k} \rightarrow T_{x_k} M^\perp \simeq \mathbb{R}^{n-m}$ to obtain $X, Y \in V_{x_k}$, $|X| \geq |Y| \geq 1$ such that $\alpha(x_k)(X, X) = \alpha(x_k)(Y, Y)$ and $\alpha(x_k)(X, Y) = 0$.

By the Gauss equation we have that

$$\begin{aligned}
K_M(x_k)(X, Y) - K_N(\varphi(x_k))(X, Y) &= \frac{\langle \alpha(x_k)(X, X), \alpha(x_k)(Y, Y) \rangle}{|X|^2 |Y|^2 - \langle X, Y \rangle^2} \\
&= \frac{|\alpha(x_k)(X, X)|^2}{|X|^2 |Y|^2} \\
&\geq \left(\frac{|\alpha(x_k)(X, X)|}{|X|^2} \right)^2 \\
&= \left| \alpha(x_k) \left(\frac{X}{|X|}, \frac{X}{|X|} \right) \right|^2
\end{aligned}$$

This implies by (32) that

$$\sup K_M - \inf K_N > \left(\left[\frac{h'(s_k)}{h(s_k)} - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1)h(s_k)} \right] \left[\frac{1}{k(t_k + 1)h(s_k)} + 1 \right]^{-1} \right)^2 > 0.$$

Therefore, $\sup K_M - \inf K_N > 0$ regardless $b = 0$ or $b < 0$. If $b < 0$ we let $k \rightarrow +\infty$ and then we have

$$\sup K_M - \inf K_N \geq \lim_{s_k \rightarrow \infty} \left[\frac{h'}{h}(s_k) \right]^2 = |b| \quad (33)$$

The case where the sequence $\{x_k\} \subset M$ remains in a compact set, we proceed as follows. Passing to a subsequence we have that $x_k \rightarrow x_\infty \in M$. Thus $t_k \rightarrow t_\infty < \infty$ and $s_k \rightarrow s_\infty < \infty$. By (24)

$$\text{Hess}_M f(x_\infty)(X, X) \leq \lim_{k \rightarrow \infty} \frac{(C_{-\Lambda^2}(t_\infty) + |\alpha(x_\infty)(X, X)|)}{k(t_\infty + 1)} = 0, \quad (34)$$

for all $X \in T_{x_0}M$. Using the expression on the right hand side of (29) we obtain for every $X \in V_{x_\infty}$

$$0 \geq \text{Hess } f(x_\infty)(X, X) \geq \left[\left(1 - \sum_{i=1}^{\ell} c_i^2 \right) \cdot \frac{h'}{h}(s_\infty) - |\alpha(X, X)| \right] \phi'_b(s_\infty).$$

There exists a sub-space $V_x \subset T_x M$ with $\dim(V_x) \geq (m - \ell) \geq 2$ such that $V \perp T\mathbb{R}^\ell$, this is equivalent to $c_i = 0$. If we take any $X \in V_{x_\infty} \subset T_{x_\infty}M$, $|X| = 1$ we have hence

$$|\alpha_{x_\infty}(X, X)| \geq \frac{h'}{h}(s_\infty) |X|^2.$$

Again, using Otsuki's Lemma and Gauss equation, we conclude that

$$\sup_M K_M - \inf_{B_N(r)} K_N \geq \frac{h'}{h}(s_\infty) > |b|. \quad (35)$$

3.3 Proof of Theorem 2.5.

We will follow the proof of Theorem 2.3 closely. Recall that g_k reaches a maximum at $x_k \in M$, $k = 1, 2, \dots$, thus so that $\triangle_M g_k(x_k) \leq 0$. Thus

$$\triangle_M f(x_k) \leq \frac{1}{k} \cdot \triangle_M (\log(\rho_L \circ \pi_L + 1) \circ \varphi(x_k)). \quad (36)$$

Using the formula (17)

$$\begin{aligned} \triangle_M (\log(\rho_L \circ \pi_L + 1) \circ \varphi(x_k)) &= \sum_{i=1}^m \text{Hess}_{N \times L} \log(\rho_{\mathbb{R}^\ell} \circ \pi_L + 1)(\varphi(x_k))(X_i, X_i) \\ &\quad + \langle \text{grad}_{N \times L} \log(\rho_L \circ \pi_L + 1), H \rangle \end{aligned} \quad (37)$$

where $H = \sum_{i=1}^m \alpha(X_i, X_i)$ is the mean curvature vector while α is the second fundamental form of the immersion φ and $\{X_i\}$ is an orthonormal basis of $T_{x_k} M$.

As before, decomposing $X \in TM$ as $X = X^N + X^L \in TN \oplus TL$ and setting $\psi(t) = \log(t + 1)$, $y_k = y(x_k)$ and $t_k = \rho_L(y_k)$ we have that the right hand side of (37)

$$\begin{aligned} \sum_{i=1}^m \text{Hess}_{N \times L} \psi \circ \rho_L \circ y(x_k)(X_i, X_i) &= \psi''(t_k) \sum_{i=1}^m |X_i^L|^2 \\ &\quad + \psi'(t_k) \sum_{i=1}^m \text{Hess}_L \rho_L(y_k)(X_i, X_i) \\ &\leq \frac{C_{-\Lambda^2}(t_k)}{(t_k + 1)} \sum_{i=1}^m |X_i^N|^2, \\ &\leq \frac{m \cdot C_{-\Lambda^2}(t_k)}{(t_k + 1)} \end{aligned} \quad (38)$$

since $\psi'' \leq 0$ and

$$\begin{aligned} \langle \text{grad}_{N \times L} \psi \circ \rho_L \circ y(x_k), H \rangle &= \psi'(t_k) \langle \text{grad} \rho_L(y_k), H \rangle \\ &\leq \frac{1}{(t_k + 1)} |H| \end{aligned} \quad (39)$$

From (37), (38) and (39) we have

$$\triangle_M \log(\rho_L(y(x_k)) + 1) \leq \frac{m \cdot C_{-\Lambda^2}(t_k) + |H|}{(t_k + 1)} \quad (40)$$

And from (36) and (40) we have that

$$\triangle_M f(x_k) \leq \frac{m \cdot C_{-\Lambda^2}(t_k) + |H|}{k(t_k + 1)} \quad (41)$$

We will compute the left hand side of (36). Recall that $f = g \circ \varphi$ and g is given by $g(z, y) = \phi_h(\rho_N(z))$, where ϕ is defined in (11). Using the formula (17) again we have

$$\triangle_M f(x_k) = \sum_{i=1}^m \text{Hess}_{N \times L} g(\varphi(x_k))(X_i, X_i) + \langle \text{grad } g, H \rangle \quad (42)$$

Consider the orthonormal basis (26) for $T_{\varphi(x_k)}(N \times L)$. Thus if $X_i \in T_{x_k} M$, $|X_i| = 1$, we can decompose

$$X_i = a_i \cdot \text{grad } \rho_N + \sum_{j=1}^{n-\ell-1} b_{ij} \cdot \partial / \partial \theta_j + \sum_{l=1}^{\ell} c_{il} \cdot \partial / \partial \gamma_l$$

with $a_i^2 + \sum_{j=1}^{n-\ell-1} b_{ij}^2 + \sum_{l=1}^{\ell} c_{il}^2 = 1$. Set $z_k = z(x_k)$ and $s_k = \rho_N(z_k)$. We have as in (27)

$$\text{Hess}_{N \times L} g(\varphi(x))(X_i, X_i) \geq (1 - \sum_{l=1}^{\ell} c_{il}^2) \cdot \phi'_h(s_k) \cdot \frac{h'}{h}(s_k) \quad (43)$$

The second term of the right hand side of (42) is the following, if $|X| = 1$,

$$\begin{aligned} \langle \text{grad } g, H \rangle &= \phi'_h(s_k) \langle \text{grad } \rho_N(z_k), H \rangle \\ &\geq -\phi'_h(s_k) |H| \end{aligned} \quad (44)$$

Therefore from (42), (43), (44) we have that,

$$\triangle_M f(x_k) \geq \left[(m - \sum_{i=1}^m \sum_{l=1}^{\ell} c_{il}^2) \cdot \frac{h'}{h}(s_k) - |H| \right] \phi'_h(s_k) \quad (45)$$

From (41) and (45) we have

$$\frac{m \cdot C_{-\Lambda^2}(t_k) + |H|}{k(t_k + 1)} \geq \triangle_M f(x_k) \geq \left[(m - \ell) \cdot \frac{h'}{h}(s_k) - |H| \right] \phi'_h(s_k) \quad (46)$$

Therefore

$$\sup_M |H| \left[\frac{1}{h(s_k) \cdot k \cdot (t_k + 1)} + 1 \right] \geq (m - \ell) \cdot \frac{h'}{h}(s_k) - \frac{m \cdot C_{-\Lambda^2}(t_k)}{h(s_k) \cdot k \cdot (t_k + 1)}$$

Letting $k \rightarrow \infty$ we have

$$\sup_M |H| \geq (m - \ell) \cdot \lim_{k \rightarrow \infty} \frac{h'}{h}(s_k).$$

If in addition, we have that $-G \leq b \leq 0$ then $(h'/h)(s) \geq C_b(s)$. The case that $b = 0$ we have $(h'/h)(s_k) \geq 1/s_k$ and $h(s_k) \geq s_k$. Since the immersion is ϕ -bounded we have $s_k^2 \leq 2 \log(t_k + 1)^{(1-\epsilon)}$. Thus for sufficient large k

$$\sup_M |H| \left[\frac{1}{s_k \cdot k \cdot (t_k + 1)} + 1 \right] \geq \frac{m - \ell}{s_k} - \frac{m \cdot C_{-\Lambda^2}(t_k)}{s_k \cdot k \cdot (t_k + 1)} > 0.$$

This shows that $\sup_M |H| > 0$.

In the case $b < 0$, we have $(h'/h)(s_k) \geq C_b(s_k) \geq \sqrt{|b|}$ and

$$\sup_M |H| \geq (m - \ell) \cdot \lim_{k \rightarrow \infty} \frac{h'}{h}(s_k) \geq \sqrt{|b|}.$$

Remark 3.4 *The statements of Theorems 2.3 and 2.5 are also true in a slightly more general situation. This is, if, instead a proper ϕ -bounded immersion, one asks a proper immersion $\varphi: M \rightarrow N \times L$ with the property*

$$\lim_{x \rightarrow \infty \text{ in } M} \frac{\phi_h(\rho_N(z(x)))}{\log(\rho_L(y(x)) + 1)} = 0,$$

where $\varphi(x) = (z(x), y(x)) \in N \times L$.

4 Omori-Yau pairs

Omori, in [18], discovered an important global maximum principle for complete Riemannian manifolds with sectional curvature bounded below. Omori's maximum principle was refined and extended by Cheng and Yau, [8], [28], [29], to Riemannian manifolds with Ricci curvature bounded below and applied to find elegant solutions to various analytic-geometric problems on Riemannian manifolds. There were others generalizations of the Omori-Yau maximum principle under more relaxed curvature requirements in [7], [10] and an extension to an all general setting by S. Pigola, M. Rigoli and A. Setti in their beautiful book [23]. There, they introduced the following terminology.

Definition 4.1 (Pigola-Rigoli-Setti) *The Omori-Yau maximum principle holds on a Riemannian manifold W if for any $u \in C^2(W)$ with $u^* := \sup_W u < \infty$, there exists a sequence of points $x_k \in W$, depending on u and on W , such that*

$$\lim_{k \rightarrow \infty} u(x_k) = u^*, \quad |\text{grad } u|(x_k) < \frac{1}{k}, \quad \Delta u(x_k) < \frac{1}{k}. \quad (47)$$

Likewise, the Omori-Yau maximum principle for the Hessian holds on W if

$$\lim_{k \rightarrow \infty} u(x_k) = u^*, \quad |\text{grad } u|(x_k) < \frac{1}{k}, \quad \text{Hess}_W u(x_k)(X, X) < \frac{1}{k} \cdot |X|^2, \quad (48)$$

for every $X \in T_{x_k} W$.

A natural and important question is, what are the Riemannian geometries the Omori-Yau maximum principle holds on? It does hold on complete Riemannian manifolds with sectional curvature bounded below holds [18], it holds on complete Riemannian manifolds with Ricci curvature bounded below [8], [28], [29]. Follows from the work of Pigola-Rigoli-Setti [23] that the Omori-Yau maximum principle holds on complete Riemannian manifolds W with Ricci curvature with strong quadratic decay,

$$\text{Ric}_W \geq -c^2 \cdot \rho_W^2 \cdot \Pi_{i=1}^k (\log^{(i)}(\rho_W + 1)), \quad \rho_W \gg 1.$$

The notion of Omori-Yau pair was formalized in [3], after the work of Pigola-Rigoli-Setti. The Omori-Yau pair is, here, described for the Laplacian and for the Hessian however, it certainly can be extended to other operators or bilinear forms.

Definition 4.2 *Let W be a Riemannian manifold. A pair (\mathcal{G}, γ) of smooth functions $\mathcal{G}: [0, +\infty) \rightarrow (0, +\infty)$, $\gamma: W \rightarrow [0, +\infty)$, $\mathcal{G} \in C^1([0, \infty))$, $\gamma \in C^2([0, \infty))$, forms an Omori-Yau pair for the Laplacian in W , if they satisfy the following conditions:*

h.1) $\gamma(x) \rightarrow +\infty$ as $x \rightarrow \infty$ in W .

$$h.2) \quad \mathcal{G}(0) > 0, \quad \mathcal{G}'(t) \geq 0 \text{ and } \int_0^{+\infty} \frac{ds}{\sqrt{\mathcal{G}(s)}} = +\infty.$$

h.3) $\exists A > 0$ constant such that $|\text{grad}_W \gamma| \leq A \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right)$ off a compact set.

h.4) $\exists B > 0$ constant such that $\Delta_W \gamma \leq B \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right)$ off a compact set.

The pair (\mathcal{G}, γ) forms an Omori-Yau pair for the Hessian if instead h.4) one has

h.5) $\exists C > 0$ constant such that $\text{Hess } \gamma \leq C \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right)$ off a compact set, in the sense of quadratic forms.

The result [23, Thm.1.9] captured the essence of the Omori-Yau maximum principle and it can be stated as follows.

Theorem 4.3 *If a Riemannian manifold M has an Omori-Yau pair (\mathcal{G}, γ) then the Omori-Yau maximum principle on it.*

The main step in the proof of Alias-Bessa-Montenegro's Theorem (Thm.1.5) and Alias-Bessa-Dajczer's Theorem (Thm.1.7) is to show that a cylindrically bounded submanifold, properly immersed into $N \times L$, with *controlled* second fundamental form or *controlled* mean curvature vector, has an Omori-Yau pair, provided L has an Omori-Yau pair. Thus, the Omori-Yau maximum principle holds on those submanifolds and their proof follows the steps of Jorge-Koutrofotis's Theorem. On the other hand, the idea behind the proof of Theorems 2.3 & 2.5 is that: the factor L has bounded sectional curvature it has a natural Omori-Yau pair (\mathcal{G}, γ) . This Omori-Yau pair together with the geometry of the factor N allows us to consider an unbounded region Ω_ϕ such that if $\phi: M \rightarrow \Omega_\phi \subset N \times L$ is an isometric immersion then there exists a function $f \in C^2(M)$, not necessarily bounded, and a sequence $x_k \in M$ satisfying $\Delta f(x_k) \leq 1/k$. We show that a properly immersed ϕ -bounded submanifold has an Omori-Yau pair for the Laplacian, provided the fiber L has an Omori-Yau pair for the Hessian. We show in Theorem 4.5 that an Omori-Yau pair for the Hessian guarantee the Omori-Yau sequence for certain unbounded functions, as this unbounded function f we are working. This leads to stronger forms of Theorem 2.3. & Theorem 2.5.

Let M, N, L be complete Riemannian manifolds of dimension $m, n - \ell$ and ℓ , with distinguished points x_0, z_0 and y_0 respectively. Suppose that $K_N^{\text{rad}} \leq -G(\rho_N)$, G satisfying (8). Let h solution of (9) and ϕ_h as in (11). Suppose in addition that L has an Omori-Yau pair for the Hessian (γ, \mathcal{G}) . Let $\Omega_{h,\gamma,\mathcal{G}}(\varepsilon) \subset N \times L$ be the region defined by

$$\Omega_{h,\gamma,\mathcal{G}}(\varepsilon) = \{(z, y) \in N \times L : \phi_h \circ \rho_N(z(x)) \leq [\psi \circ \gamma(y(x))]^{1-\varepsilon}\},$$

where $\psi(t) = \log \left(\int_0^t \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right)$. In this setting we have the following result.

Theorem 4.4 *Let $\phi: M \rightarrow N \times L$ be a properly immersed submanifold such that $\phi(M \setminus K) \subset \Omega_{h,\gamma,\mathcal{G}}(\varepsilon)$ for some compact $K \subset M$ and positive $\varepsilon \in (0, 1)$.*

1. *If $K_N^{\text{rad}} \leq -G \leq b \leq 0$ and the codimension satisfies $n - m \leq m - \ell - 1$ then*

$$\sup_M K_M \geq |b| + \inf_N K_N. \quad (49)$$

With strict inequality $\sup_M K_M > \inf_N K_N$ if $b = 0$.

2. If $m \geq \ell + 1$ then

$$\sup_M |H| \geq (m - \ell) \cdot \inf_{r \in [0, \infty)} \frac{h'}{h}(r). \quad (50)$$

If $-G \leq b \leq 0$ then

$$\sup_M |H| \geq (m - \ell) \cdot \sqrt{|b|}. \quad (51)$$

With strict inequality $\sup_M |H| > 0$ if $b = 0$.

Assume without loss of generality that there exists $x_0 \in M$ such that $\varphi(x_0) = (z_0, y_0) \in N \times L$. As before, $\varphi(x) = (z(x), y(x))$ and $g, p: N \times L \rightarrow \mathbb{R}$ given by $g(z, y) = \phi_h(\rho_N(z)) + \psi(\gamma(y))$, $p(z, y) = \psi(\gamma(y))$.

For each $k \in \mathbb{N}$, let $g_k: M \rightarrow \mathbb{R}$ given by $g_k(x) = g \circ \varphi(x) - p \circ \varphi(x)/k$. Observe that $g_k(x_0) = 1$ and for $\rho_M(x) \gg 1$, we have that $g_k(x) < 0$. This implies that g_k has a maximum at a point x_k , yielding in this way a sequence $\{x_k\} \subset M$ such that $\text{Hess}_M g_k(x_k) \leq 0$ in the sense of quadratic forms. Proceeding as in the proof of Theorem 2.3 we have that for $X \in T_{x_k} M$,

$$\text{Hess}_M g \circ \varphi(x_k)(X, X) \leq \frac{1}{k} \text{Hess}_M p \circ \varphi(x_k)(X, X). \quad (52)$$

We have to compute both terms of this inequality. Considering once more the orthonormal basis (26) for $T_{\varphi(x_k)}(N \times L)$ we can decompose, $X \in T_{x_k} M$, $|X| = 1$, (after identifying X with $d\varphi X$), as

$$X = a \cdot \text{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial / \partial \theta_j + \sum_{i=1}^{\ell} c_i \cdot \partial / \partial \gamma_i$$

with $a^2 + \sum_{j=1}^{n-\ell-1} b_j^2 + \sum_{i=1}^{\ell} c_i^2 = 1$. Setting $s_k = \rho_N(z(x_k))$, $t_k = \gamma(y(x_k))$, we have as in (29),

$$\begin{aligned} \text{Hess}_M g \circ \varphi(x_k)(X, X) &= \text{Hess}_{N \times L} g(\varphi(x_k))(X, X) + \langle \text{grad}_{N \times L} g, \alpha(X, X) \rangle \\ &\geq \left[\left(1 - \sum_{i=1}^{\ell} c_i^2\right) \cdot \frac{h'}{h}(s_k) - |\alpha(X, X)| \right] \phi'_b(s_k) \end{aligned} \quad (53)$$

$$\begin{aligned}
\text{Hess}_M p \circ \varphi(x_k)(X, X) &= \text{Hess}_{N \times L} p(\varphi(x_k))(X, X) + \langle \text{grad}_{N \times L} p, \alpha(X, X) \rangle \\
&= \psi''(t_k) \langle X, \text{grad}_L \gamma \rangle^2 + \psi'(t_k) \text{Hess}_L \gamma(X, X) \\
&\quad + \psi'(t_k) \langle \text{grad}_L \gamma, \alpha(X, X) \rangle \\
&\leq \psi'(t_k) (\text{Hess}_L \gamma(X, X) + |\text{grad}_L \gamma| \cdot |\alpha(X, X)|) \quad (54) \\
&\leq \frac{\left[\sqrt{\mathcal{G}(\gamma(t_k))} \left(\int_0^{t_k} \frac{ds}{\sqrt{\mathcal{G}(\gamma(s))}} + 1 \right) \right] (C + A \cdot |\alpha(X, X)|)}{\sqrt{\mathcal{G}(\gamma(t_k))} \left(\int_0^{t_k} \frac{ds}{\sqrt{\mathcal{G}(\gamma(s))}} + 1 \right)} \\
&= C + A \cdot |\alpha(X, X)|,
\end{aligned}$$

since $\psi'' \leq 0$. Taking in consideration the bounds (53) & (54), the inequality (52) yields, $(\phi'(s) = h(s))$,

$$\left[\frac{A}{k \cdot h(s_k)} + 1 \right] |\alpha(X, X)| \geq (1 - \sum_{i=1}^{\ell} c_i^2) \frac{h'}{h}(s_k) - \frac{C}{k \cdot h(s_k)}. \quad (55)$$

Under the hypotheses of item 1. we have that $(h'/h)(s) \geq C_b(s) > \sqrt{|b|}$ and $h(s) \rightarrow \infty$ as $s \rightarrow \infty$. Moreover, there exists a subspace $V_{x_k} \subset T_{x_k} M$, $\dim V_{x_k} \geq 2$, such that if $X \in V_{x_k}$ then $X = a \cdot \text{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial / \partial \theta_j$. Therefore, for $X \in V_{x_k}$, $|X| = 1$, we have for $k \gg 1$.

$$\begin{aligned}
\left[\frac{A}{k \cdot h(s_k)} + 1 \right] |\alpha(X, X)| &\geq \frac{h'}{h}(s_k) - \frac{C}{k \cdot h(s_k)} \\
&> |b| - \frac{C}{k \cdot h(s_k)} \\
&> 0.
\end{aligned} \quad (56)$$

The proof follows exactly the steps of the proof of Theorem 2.3 and we obtain that $\sup_M K_M \geq |b| + \inf_N K_N$ if $b < 0$ and $\sup_M K_M > \inf_N K_N$ if $b = 0$.

To prove item 2., take an orthonormal basis $X_1, \dots, X_q, \dots, X_m \in T_{x_k} M$,

$$X_q = a_q \cdot \text{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_{jq} \cdot \partial / \partial \theta_j + \sum_{i=1}^{\ell} c_{iq} \cdot \partial / \partial \gamma_i$$

with $a_q^2 + \sum_{j=1}^{n-\ell-1} b_{jq}^2 + \sum_{i=1}^{\ell} c_{iq}^2 = 1$. Tracing the inequality (55) to obtain

$$\begin{aligned} \left[\frac{A}{k \cdot h(s_k)} + 1 \right] |H| &\geq (m - \sum_{q=1}^m \sum_{i=1}^{\ell} c_{iq}^2) \frac{h'(s_k)}{h(s_k)} - \frac{C}{k \cdot h(s_k)} \\ &\geq (m - \ell) C_b(s_k) - \frac{C}{k \cdot h(s_k)} \\ &> 0 \end{aligned} \quad (57)$$

for $k \gg 1$. If $b = 0$ then $C_b(s) = 1/s$ then, coupled with the estimate $h(s) \geq s\sqrt{s}$, see [6], we deduce that $\sup_M |H| > 0$. And if $b < 0$ then $C_b(s) \geq \sqrt{|b|} > 0$, then letting $k \rightarrow \infty$ we have $\sup_M |H| \geq (m - \ell) \sqrt{|b|} > 0$ if $b < 0$. We can see these curvature estimates as geometric applications of the following extension of the Pigola, Rigoli, Setti [23, Thm.1.9].

Theorem 4.5 *Let W be a complete Riemannian manifold with an Omori-Yau pair (\mathcal{G}, γ) for the Hessian (Laplacian). If $u \in C^2(W)$ satisfies $\lim_{x \rightarrow \infty} \frac{u(x)}{\psi(\gamma(x))} = 0$, where*

$\psi(t) = \log \left(\int_0^t \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right)$, then there exist a sequence $x_k \in M$, $k \in \mathbb{N}$ such that

$$|\text{grad}_w u|(x_k) \leq \frac{A}{k}, \quad \text{Hess}_w u(x_k) \leq \frac{C}{k} \quad (\Delta_w u(x_k) \leq \frac{B}{k}) \quad (58)$$

If $u^ = \sup_M u < \infty$ then $u(x_k) \rightarrow u^*$. The constants A , B and C come from the Omori-Yau pair (\mathcal{G}, γ) , see Definition 4.2.*

This result above should be compared with [21, Cor. A1.], due to Pigola, Rigoli, and Setti where they proved an Omori-Yau for quite general operators, applicable to certain unbounded functions with growth to infinity faster than ours. However, we could replace the distance function of their result by an Omori-Yau pair. It would be interesting to understand these facts.

Assume that the Omori-Yau pair (\mathcal{G}, γ) is for the Hessian. The case of the Laplacian is similar. Fix a point $x_0 \in M$ such that $\gamma(x_0) > 0$ and define for each $k \in \mathbb{N}$, $g_k: M \rightarrow \mathbb{R}$ by $g_k(x) = u(x) - \frac{1}{k} \psi(\gamma(x)) + 1 - u(x_0) - \frac{1}{k} \psi(\gamma(x_0))$. We have that $g_k(x_0) = 1$ and $g_k(x) \leq 0$ for $\rho_w(x) = \text{dist}_w(x_0, x) \gg 1$. Thus there is a point x_k such that g_k reaches a maximum. This way we find a sequence $x_k \in M$ such that

for all $X \in T_{x_k} W$

$$\begin{aligned}
\text{Hess}_w u(X, X) &\leq \frac{1}{k} \text{Hess}_w \psi(\gamma)(X, X) \\
&= \frac{1}{k} [\psi''(\gamma) \langle \text{grad}_w \gamma, X \rangle^2 + \psi'(\gamma) \text{Hess}_w \gamma(X, X)] \\
&\leq \frac{1}{k} \left[\frac{1}{\sqrt{\mathcal{G}(\gamma)}} \frac{1}{\left(\int_0^\gamma \frac{ds}{\mathcal{G}(s)} + 1 \right)} C \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\mathcal{G}(s)} + 1 \right) \right] |X|^2 \\
&= \frac{C}{k} |X|^2.
\end{aligned}$$

We used that $\psi'' \leq 0$ and $\text{Hess}_w \gamma(X, X) \leq C \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\mathcal{G}(s)} + 1 \right)$.

$$\begin{aligned}
|\text{grad}_w u| &= \frac{1}{k} |\text{grad}_w \psi(\gamma)| \\
&\leq \frac{1}{k} \left[\frac{1}{\sqrt{\mathcal{G}(\gamma)}} \frac{1}{\left(\int_0^\gamma \frac{ds}{\mathcal{G}(s)} + 1 \right)} A \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\mathcal{G}(s)} + 1 \right) \right] \\
&\leq \frac{A}{k}.
\end{aligned}$$

4.1 Omori-Yau pairs and warped products

Let (N, g_N) and (L, g_L) be complete Riemannian manifolds of dimension $n - \ell$ and ℓ respectively and $\xi: L \rightarrow \mathbb{R}_+$ be a smooth function. Let $\varphi: M \rightarrow L \times_\xi N$ be an isometric immersion into the warped product $L \times_\xi N = (L \times N, ds^2 = g_L + \xi^2 g_N)$. The immersed submanifold $\varphi(M)$ is cylindrically bounded if $\pi_N(\varphi(M)) \subset B_N(r)$, where $\pi_N: L \times N \rightarrow N$ is the canonical projection in the N -factor and $B_N(r)$ is a regular geodesic ball of radius r of N . Alías and Dajczer in the proof of [4, Thm.1], showed that if φ is proper in $L \times N$ then the existence of an Omori-Yau pair for the Hessian in L induces an Omori-Yau pair for the Laplacian on M provided the mean curvature $|H|$ is bounded. We can prove a slight extension of this result.

Lemma 4.6 *Let $\varphi: M \rightarrow L \times_\xi N$ be an isometric immersion, proper in the first entry, where L carries an Omori-Yau pair (\mathcal{G}, γ) for the Hessian, $\xi \in C^\infty(L)$ is a positive function satisfying*

$$|\text{grad} \log \xi(y)| \leq \ln \left(\int_0^{\gamma(y)} \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right). \quad (59)$$

Letting $\varphi(x) = (y(x), z(x))$ and if

$$|H(\varphi(x))| \leq \ln \left(\int_0^{\gamma(y(x))} \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right), \quad (60)$$

then M has an Omori-Yau pair for the Laplacian. In particular, M holds the Omori-Yau maximum principle for the Laplacian.

The idea of the proof is presented in [4] and therefore will try to follow the same notation to simplify the demonstration. Let (\mathcal{G}, γ) the Omori-Yau pair for the Hessian of L . Assume w.l.o.g. that M is non-compact and denote $\varphi(x) = (y(x), z(x))$. Define $\Gamma(y, z) = \gamma(y)$ and define $\vartheta(x) = \Gamma \circ \varphi = \gamma(y(x))$. We will show that (\mathcal{G}, ϑ) is an Omori-Yau pair for the Laplacian in M . Indeed, let $q_k \in M$ a sequence such that $q_k \rightarrow \infty$ in M as $k \rightarrow +\infty$. Since φ is proper in the first entry, we have that $y(q_k) \rightarrow \infty$ in L . Since $\vartheta(q_k) = \gamma(y(q_k))$ we have $\vartheta(q_k) \rightarrow \infty$ as $q_k \rightarrow \infty$ in M .

We have that

$$\text{grad}_{L \times_{\xi} N} \Gamma(z, y) = \text{grad}_L \gamma(z). \quad (61)$$

Since $\xi = \Gamma \circ \varphi$, we obtain at $\varphi(q)$

$$\begin{aligned} \text{grad}_{L \times_{\xi} N} \Gamma &= (\text{grad}_{L \times_{\xi} N} \Gamma)^T + (\text{grad}_{L \times_{\xi} N} \Gamma)^{\perp} \\ &= \text{grad}_M \xi + (\text{grad}_{L \times_{\xi} N} \Gamma)^{\perp}. \end{aligned}$$

By hypothesis we have

$$\begin{aligned} |\text{grad}_M \xi|(q) &\leq |\text{grad}_{L \times_{\xi} N} \Gamma|(\varphi(q)) = |\text{grad}_L \gamma|(y(q)) \\ &\leq \sqrt{\mathcal{G}(\gamma(y(q)))} \left(\int_0^{\gamma(y(q))} \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right) \end{aligned}$$

out of a compact subset of M .

Let $T, S \in TL, X, Y \in TN$ and $\nabla^{L \times_{\xi} N}, \nabla^L$ and ∇^N be the Levi-Civita connections of the metrics $ds^2 = g_L + \xi^2 g_N$, g_L and g_N respectively. It is easy to show that $\nabla_S^{L \times_{\xi} N} T = \nabla_S^L T$ and $\nabla_X^{L \times_{\xi} N} T = \nabla_T^{L \times_{\xi} N} X = T(\eta)X$ where $\eta = \log \xi$. Therefore,

$$\begin{aligned} \nabla_T^{L \times_{\xi} N} \text{grad}_{L \times_{\xi} N} \Gamma &= \nabla_T^L \text{grad}_L \gamma \\ \nabla_X^{L \times_{\xi} N} \text{grad}_{L \times_{\xi} N} \Gamma &= \text{grad}_L \gamma(\eta)X. \end{aligned}$$

Hence,

$$\text{Hess}_{L \times_{\xi} N} \Gamma(T, S) = \text{Hess}_L \gamma(T, S), \text{Hess}_{L \times_{\xi} N} \Gamma(T, X) = 0$$

$$\text{Hess}_{L \times_{\xi} N} \Gamma(X, Y) = \langle \text{grad}_L \eta, \text{grad}_L \gamma \rangle \langle X, Y \rangle.$$

For any unit vector $e \in T_q M$, decompose $e = e^L + e^N$, where $e^L \in T_{y(q)} L$ and $e^N \in T_{z(q)} N$. Then we have at $\varphi(q)$

$$\text{Hess}_{L \times_{\xi} N} \Gamma(e, e) = \text{Hess}_L \gamma(y(q))(e^L, e^L) + \langle \text{grad}_L \gamma, \text{grad}_L \eta \rangle(y(q)) |e^N|^2.$$

On the other hand, $\text{Hess}_M \xi(q)(e, e) = \text{Hess}_{L \times_{\xi} N} \Gamma(e, e) + \langle \text{grad}_{L \times_{\xi} N} \Gamma, \alpha(e, e) \rangle$. Therefore,

$$\begin{aligned} \text{Hess}_M \xi(q)(e, e) &= \text{Hess}_L \gamma(e^L, e^L) + \langle \text{grad}_L \gamma, \text{grad}_L \eta \rangle(z(q)) |e^P|^2 \\ &\quad + \langle \text{grad}_L \gamma, \alpha(e, e) \rangle. \end{aligned} \tag{62}$$

However,

$$\text{Hess}_L \gamma \leq \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right), \tag{63}$$

out of a compact subset of L . By hypothesis, see (59),

$$\begin{aligned} \langle \text{grad}_L \gamma, \text{grad}_L \eta \rangle(y(q)) &\leq |\text{grad}_L \gamma| \cdot |\text{grad}_L \eta| \\ &\leq \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right) \ln \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right) \end{aligned} \tag{64}$$

Considering (63), (64) and (62) we have that (off a compact set)

$$\begin{aligned} \text{Hess}_M \xi(q)(e, e) &\leq C \cdot \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right) \ln \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right) \\ &\quad + \langle \text{grad}_L \gamma, \alpha(e, e) \rangle, \end{aligned}$$

for some positive constant C . Thus, by (60) it follows that

$$\Delta \gamma \leq B \sqrt{\mathcal{G}(\gamma)} \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right) \ln \left(\int_0^\gamma \frac{ds}{\sqrt{\mathcal{G}(s)}} + 1 \right)$$

for some positive constant B . Concluding that (\mathcal{G}, ξ) is an Omori-Yau pair for the Laplacian in M . The proof of [4, Thm.1] coupled with Lemma 4.6 allows us to state the following technical extension of Alias-Dajczer's Theorem [4, Thm.1].

Theorem 4.7 (Alias-Dajczer) *Let $\varphi: M \rightarrow L \times_{\xi} N$ be an isometric immersion, proper in the first entry, where L carries an Omori-Yau pair (G, γ) for the Hessian, $\xi \in C^{\infty}(L)$ is a positive function satisfying*

$$|\text{grad } \log \xi(y)| \leq \ln \left(\int_0^{\gamma(y)} \frac{ds}{\sqrt{G(s)}} + 1 \right). \quad (65)$$

Letting $\varphi(x) = (y(x), z(x))$ and if

$$|H(\varphi(x))| \leq \ln \left(\int_0^{\gamma(y(x))} \frac{ds}{\sqrt{G(s)}} + 1 \right). \quad (66)$$

Suppose that $\varphi(M) \subset \{(y, z) : y \in L, z \in B_N(r)\}$ then

$$\sup_M \xi |H| \geq (m - \ell) C_b(r),$$

where $b = \sup_{B_N(r)} K_N^{\text{rad}}$.

Remark 4.8 *The Theorems 2.3 & 2.5 should have versions for ϕ -bounded submanifold of warped product $L \times_{\xi} N$. Specially interesting should be the Jorge-Koutrofotis Theorem in this setting. We leave to the interested reader to pursue it.*

As a last application of Theorem 4.5, let $N^{n+1} = I \times_{\xi} P^n$ the product manifold endowed with the warped product metric, $I \subset \mathbb{R}$ is a open interval, P^n is a complete Riemannian manifold and $\xi: I \rightarrow \mathbb{R}_+$ is a smooth function. Given an isometrically immersed hypersurface $\varphi: M^n \rightarrow N^{n+1}$, define $h: M^n \rightarrow I$ the $C^{\infty}(M^n)$ height function by setting $h = \pi_I \circ \varphi$, where $\pi_I: I \times P \rightarrow I$ is a projection. This result below is a technical extension of [5, Thm.7] its proof is exactly as there, we just relaxed the hypothesis guaranteeing an Omori-Yau sequence.

Theorem 4.9 *Let $\varphi: M^n \rightarrow N^{n+1}$ be an isometrically immersed hypersurface. If M^n has an Omori-Yau pair (G, γ) for the Laplacian and the height function h satisfies $\lim_{x \rightarrow \infty} \frac{h(x)}{\psi(\gamma(x))} = 0$ then*

$$\sup_{M^n} |H| \geq \inf_{M^n} \mathcal{H}(h), \quad (67)$$

with H being the mean curvature and $\mathcal{H}(t) = \frac{\rho'(t)}{\rho(t)}$.

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